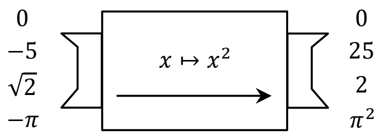
Chapter 1 Functions

1. Functions and their graphs

In the subject of calculus, the central and fundamental objects that we study are **functions**. Before formally defining what a function is, let's start with an informal understanding of the concept.

Remark 1.1 A function can be understood as a "**machine**". Every time if we give it an **input**, the machine processes the input according to some rules, and then gives us a definite **output**. As an example, we consider the following "machine".



Whenever we give it a real number x as an input, it returns us the output x^2. Such a machine is going to be represented by a formula

$$f(x) = x^2.$$

Here the symbol f is the name of the machine, the symbol x denotes an input, and the symbol $f(x)$ denotes the corresponding output. There are also certain characteristics of the machine as in

- ⦿ **what kinds of inputs are allowable** (as an example, you cannot input a set A into the above machine; the "square of a set" is not a well-defined idea),
- ⦿ **what kinds of outputs are expected** (if you input a real number x, then you expect that the output x^2 is also a real number), and
- ⦿ **what kinds of outputs are actually achievable** (if you input a real number x, then the output real number x^2 can only be non-negative; negative real numbers are not actually achievable).

Definition 1.2 Let A and B be sets. A ***function*** (or a ***map***, or a ***mapping***, or a ***transformation***) f ***from A to B***, denoted by

$$f: A \to B,$$

is a relation that connects the elements of A and the elements of B. <u>Given each element $x \in A$, this relation assigns to it a unique element $y \in B$</u>. In case y is assigned to x by the function f, we may say that "f sends x to y", and we write

$$y = f(x).$$

- ⦿ The set A is called the ***domain*** of f and the set B is called the ***codomain*** of f.
- ⦿ The ***range*** (or ***image***) of a function $f: A \to B$ is the set that contains precisely all the elements in B that are <u>actually</u> assigned to some element in A. In symbols, the range of f is the set

$$\{f(x):\ x \in A\}.$$

In particular, the range of f is part (or possibly all) of the codomain of f.

Remark 1.3 In the "machine analogy" of functions, each input can only be processed by the machine to give **one and only one output**.

- The domain is the set of all **"allowable inputs"**;
- The codomain describes **what kind of objects the "outputs" are**; and
- The range is the set of all **"achievable outputs"**.

Example 1.4 The area A of a circular disk is a function of the radius r of the disk. This function is $A: (0, +\infty) \to \mathbb{R}$ defined by

$$A(r) = \pi r^2.$$

Example 1.5 Suppose that you are buying a fish in a supermarket. Then the price $\$P$ of the fish is a function of the weight (mass?) w kg of the fish. This function is $P: (0, +\infty) \to \mathbb{R}$ defined by

$$P(w) = 50w$$

if the particular kind of fish is being sold at $\$50$ per kg.

Remark 1.6 In the two examples above, note that it is perfectly fine to input a non-positive number into the formulas $A(r) = \pi r^2$ or $P(w) = 50w$. However, it does not make sense to talk about a disk with zero or negative radius, or a fish with zero or negative weight; so instead of $\mathbb{R}$, we take $(0, +\infty)$ to be the domain of A and of P.

Example 1.7 Suppose that you are taking the course MATH1013. Then the letter grade g you finally obtain is a function of the total score x you get. This function is

$$g: [0, 100] \to \{\mathrm{A+}, \mathrm{A}, \mathrm{A-}, \mathrm{B+}, \mathrm{B}, \mathrm{B-}, \mathrm{C+}, \mathrm{C}, \mathrm{C-}, \mathrm{D}, \mathrm{F}\}$$

whose defining formula $g(x)$ is currently unknown. (Of course $g(0) = \mathrm{F}$ and $g(100) = \mathrm{A+}$.)

Example 1.8 Let S be the set of all ***finite sets*** (i.e. sets which contains finitely many elements). We define a function $n: S \to \mathbb{R}$ by

$$n(A) = \text{the number of elements in } A.$$

This function n satisfies that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \qquad \text{for every finite sets } A \text{ and } B.$$

In MATH1013 and MATH1014 we usually deal with functions whose domains, codomains and ranges are **<u>sets of real numbers</u>**.

Example 1.9 Let $f: \{1, 2, 4, 7\} \to \mathbb{R}$ be the function defined by

$$f(1) = 1, \qquad f(2) = 3, \qquad f(4) = 1 \qquad \text{and} \qquad f(7) = -6.$$

The domain of f is $\{1, 2, 4, 7\}$. The codomain of f is $\mathbb{R}$, while the range of f is $\{1, 3, -6\}$ (which is indeed part of the codomain of f).

Example 1.10 Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x^2.$$

The domain of f is $\mathbb{R}$. Although the codomain $\mathbb{R}$ consists of all the real numbers, it can be easily seen that

- ⊙ Each real number "input" x can only give a **non-negative number "output"** x^2, and
- ⊙ Each non-negative "output" y is indeed "achievable" by some real "input", namely $\sqrt{y}$ and $-\sqrt{y}$, because

$$f(\sqrt{y}) = y \qquad \text{and} \qquad f(-\sqrt{y}) = y.$$

So the range of f is $[0, +\infty)$, the set of all non-negative real numbers (which is indeed part of the codomain of f).

Remark 1.11 In all the previous examples, the domains of the given functions are already specified. In case we are given just a defining formula $f(x)$ without specifying the domain of f, we usually take the domain of f to be its ***natural domain***, which is the **largest possible set of real numbers** on which the formula $f(x)$ is "valid" (in MATH1013, this "valid" means "well-defined as a real number").

(i) To find the **natural domain** of a function f with the defining formula $f(x)$ given, it is often useful to recall **arithmetic facts** for real numbers that you have already learnt in high school:
- ⊙ Any real number divided by zero is undefined.
- ⊙ The product of two negative numbers is positive.
- ⊙ The n^{th} root of a positive number is always positive.
- ⊙ The n^{th} root of a negative number is $\begin{cases} \text{negative} & \text{if } n \text{ is odd} \\ \text{not a real number} & \text{if } n \text{ is even} \end{cases}$.

(ii) To find the **range** of a function f, we basically try to answer the question

"<u>for what values of y</u> does the equation $y = f(x)$ have solution in the domain of f?".

There is no general method to obtain an answer to this question, and it depends very much on the defining formula.

Example 1.12 Find the (natural) domain and the range of each of the following functions:

(a) $f(x) = \sqrt{x+2}$

(b) $g(x) = \sqrt{x^2+5}$

(c) $h(x) = \dfrac{x+5}{x-2}$

Solution:

(a) The expression $\sqrt{x+2}$ is well-defined if and only if $x + 2 \geq 0$, i.e. $x \geq -2$. So the natural domain of f is $[-2, +\infty)$.

On the other hand, given any $x \in [-2, +\infty)$, the number $f(x) = \sqrt{x+2}$ can take any value in $[0, +\infty)$. Therefore the range of f is $[0, +\infty)$.

(b) The expression $\sqrt{x^2+5}$ is well-defined if and only if $x^2+5 \geq 0$, which holds for every real number x. So the natural domain of g is $\mathbb{R}$.

On the other hand, given any real number x, the number $g(x) = \sqrt{x^2+5}$ can take any value in $[\sqrt{5}, +\infty)$ (note that the expression under the square-root is always ≥ 5). Therefore the range of g is $[\sqrt{5}, +\infty)$.

(c) The expression $\frac{x+5}{x-2}$ is well-defined if and only if $x-2 \neq 0$, i.e. $x \neq 2$. So the natural domain of h is $\mathbb{R} \setminus \{2\}$ or $(-\infty, 2) \cup (2, +\infty)$.

On the other hand, given any $x \in \mathbb{R} \setminus \{2\}$, the number $h(x) = \frac{x+5}{x-2} = 1 + \frac{7}{x-2}$ can take any value in $\mathbb{R} \setminus \{1\}$ (note that the expression $\frac{7}{x-2}$ is always non-zero). Therefore the range of h is $\mathbb{R} \setminus \{1\}$ or $(-\infty, 1) \cup (1, +\infty)$.

A function can be visually presented using its **graph** in the coordinate plane.

Definition 1.13 Let A be a set of real numbers and $f: A \to \mathbb{R}$ be a function. The ***graph*** of f is the set of points in the coordinate plane defined by

$$\{(x, f(x)) : x \in A\}.$$

We also say that the graph of f is described by the equation $y = f(x)$.

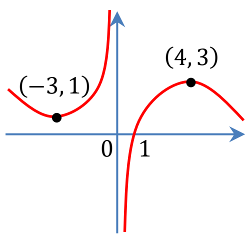
Remark 1.14 Here are some facts about the graph of a function f:

(i) ("Vertical line test") Each vertical line in the coordinate plane must intersect the graph of f at **one point at most**.

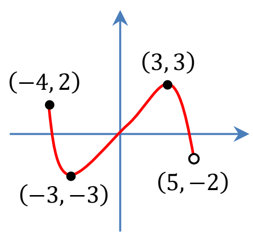
(ii) The **domain** of f consists of all those real numbers a such that the **vertical line** $x = a$ intersects the graph of f at exactly one point.

(iii) The **range** of f consists of all those real numbers b such that the **horizontal line** $y = b$ intersects the graph of f at one point at least.

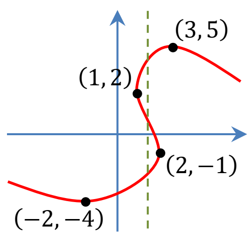
Example 1.15 Determine whether each of the following is the graph of a function. If the answer is yes, then also identify the domain and the range of the function that it represents:



Graph of a function, whose domain is $\mathbb{R} \setminus \{0\}$ and range is $\mathbb{R}$



Graph of a function, whose domain is $[-4, 5)$ and range is $[-3, 3]$



Not the graph of a function

The first and perhaps the easiest kind of functions that we will study are **polynomials**. Their natural domain is $\mathbb{R}$.

Definition 1.16 Let n be a non-negative integer. A ***polynomial*** (in a variable x) is an algebraic expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \dots, a_n$ are numbers.

- For each $k \in \{0, 1, \dots, n\}$, the number a_k is called the ***coefficient of*** x^k in the polynomial f.
- The ***degree*** of the polynomial f is $\deg f = n$, if the leading coefficient a_n is non-zero.
- A polynomial of degree 1 (resp. 2, 3) is called a ***linear*** (resp. ***quadratic***, ***cubic***) ***polynomial***.

A polynomial f can be regarded as a function $f: \mathbb{R} \to \mathbb{R}$ if all its coefficients are real numbers. A ***real root*** of a polynomial f is a real number a such that $f(a) = 0$.

Recall the facts about **polynomials and their graphs** that you have learnt in high school, including:

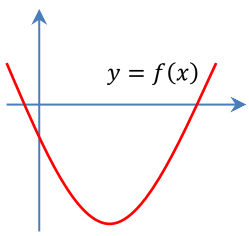
- Slope, inclination angle and intercepts of the graph of a linear polynomial $f(x) = mx + c$
- Slopes of parallel lines and perpendicular lines
- Axis of symmetry, intercepts, vertex coordinates and extremum values of a quadratic polynomial $f(x) = ax^2 + bx + c$
- Ranges of polynomials of odd degree and even degree

Example 1.17 Determine the signs of a, b and c in the following graphs of the function

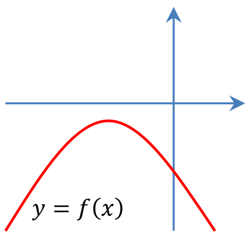
$$f(x) = ax^2 + bx + c,$$

i.e. whether a, b and c are positive or negative or zero.

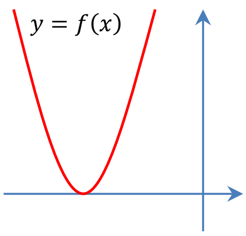
(a)



(b)



(c)



Solution:

Recall that from the graph of the quadratic function $f(x) = ax^2 + bx + c$,

- a is positive if the parabola opens upward and is negative if the parabola opens downward,
- $b = -2ah$ where h is the x-coordinate of the vertex of the parabola, and
- c is the y-intercept of the parabola.

So according to the above graphs, we have

(a) $a > 0$ and $b < 0$ and $c < 0$

(b) $a < 0$ and $b < 0$ and $c < 0$

(c) $a > 0$ and $b > 0$ and $c > 0$

Example 1.18 Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = 2x^2 - 8x + 6.$$

Find the natural domains and the ranges of the following functions.

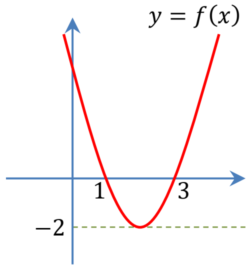
(a) $g(x) = \sqrt{f(x)}$

(b) $h(x) = \dfrac{1}{f(x)}$

Solution:

We first rewrite the quadratic function f into various different forms:

$$\begin{aligned} f(x) &= 2x^2 - 8x + 6 \\ &= 2(x-1)(x-3) \\ &= 2(x-2)^2 - 2, \end{aligned}$$



so the graph of f is a parabola which opens upward and intersects the x-axis at 1 and at 3, and is easily sketched as in the diagram.

(a) ⊙ The expression $\sqrt{f(x)}$ is well-defined if and only if $f(x) \geq 0$, i.e.

$$2(x-1)(x-3) \geq 0.$$

This holds if and only if $x \leq 1$ or $x \geq 3$. So g has natural domain $(-\infty, 1] \cup [3, +\infty)$.

⊙ According to the graph of f, we see that on the domain of g, f takes every value in $[0, +\infty)$. The range of $g = \sqrt{f}$ is also $[0, +\infty)$.

(b) ⊙ The expression $\frac{1}{f(x)}$ is well defined if and only if $f(x) \neq 0$, i.e.

$$2(x-1)(x-3) \neq 0.$$

This holds if and only if $x \neq 1$ and $x \neq 3$. So h has natural domain $\mathbb{R} \setminus \{1, 3\}$, or equivalently $(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$.

⊙ Noting that the minimum value of f is -2, we see that on the domain of h, f takes every value in $[-2, 0) \cup (0, +\infty)$. So the range of $h = \frac{1}{f}$ is $\left(-\infty, -\frac{1}{2}\right] \cup (0, +\infty)$.

The following is an important theorem regarding the roots of a polynomial.

Theorem 1.19 *Let f be a polynomial of degree n. Then f has at most n real roots.*

Example 1.20 Since

$$f(x) = x^3 + 3x - 1$$

is a polynomial of degree 3, it has at most 3 real roots.

2. Operations on functions and graphs

We try to build new functions from old ones that we already know, e.g. polynomials.

Definition 1.21 Let f and g be two real-valued functions. Then the ***sum*** $f+g$, the ***difference*** $f-g$, the ***product*** fg and the ***quotient*** $\frac{f}{g}$ are defined as follows:

- $(f+g)(x) \coloneqq f(x)+g(x)$ for every x,
- $(f-g)(x) \coloneqq f(x)-g(x)$ for every x,
- $(fg)(x) \coloneqq f(x)g(x)$ for every x, and
- $\dfrac{f}{g}(x) \coloneqq \dfrac{f(x)}{g(x)}$ for every x which satisfies $g(x) \neq 0$.

Think: How are the domains of these new functions related to those of f and g?

Applying the quotient operation to polynomials, we obtain a new kind of functions called **rational functions**. Because dividing by the number zero is "illegal", the natural domain of a rational function is the set of all real numbers **except the roots of the polynomial in the denominator**.

Definition 1.22 A ***rational function*** is a quotient of polynomials, i.e. a function f of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials and q is not the zero polynomial.

Apart from the four usual arithmetic operations, there is a "fifth arithmetic operation" on functions.

Definition 1.23 Let f and g be two functions. Then the ***composition*** $f \circ g$ is the function defined by

$$(f \circ g)(x) \coloneqq f\big(g(x)\big) \qquad \text{for every } x.$$

Think: How is the domain of $f \circ g$ related to those of f and g?

Example 1.24 Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = x+1.$$

What are the functions $f \circ g$ and $g \circ f$? Are they the same as each other?

Solution:
For every $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f\big(g(x)\big) = f(x+1) = (x+1)^2 = x^2+2x+1,$$

and

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^2) = x^2+1.$$

It is clear that $f \circ g \neq g \circ f$.

Example 1.25 Consider the rational functions

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = \frac{x+1}{x-2}$$

defined on their natural domains respectively. Find the functions $f \circ f$, $f \circ g$ and $g \circ f$.

Solution: [Note that the task includes not only finding the formulas of the three functions, but also identifying their domains.]

First note that the domain of f is $\mathbb{R} \setminus \{0\}$, and the domain of g is $\mathbb{R} \setminus \{2\}$.

- For each $x \in \mathbb{R} \setminus \{0\}$, we have

$$(f \circ f)(x) = f\big(f(x)\big) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x.$$

Therefore $f \circ f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by $(f \circ f)(x) = x$.

- For each $x \in \mathbb{R} \setminus \{2\}$, we have

$$(f \circ g)(x) = f\big(g(x)\big) = f\left(\frac{x+1}{x-2}\right) = \frac{1}{\frac{x+1}{x-2}} = \frac{x-2}{x+1} \qquad \text{except when } x = -1.$$

Therefore $f \circ g : \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R}$ is given by $(f \circ g)(x) = \frac{x-2}{x+1}$.

- For each $x \in \mathbb{R} \setminus \{0\}$, we have

$$(g \circ f)(x) = g\big(f(x)\big) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x} + 1}{\frac{1}{x} - 2} = \frac{1+x}{1-2x} \qquad \text{except when } x = \frac{1}{2}.$$

Therefore $g \circ f : \mathbb{R} \setminus \left\{0, \frac{1}{2}\right\} \to \mathbb{R}$ is given by $(g \circ f)(x) = \frac{1+x}{1-2x}$.

Remark 1.26 In the above example, we note the following important issue.

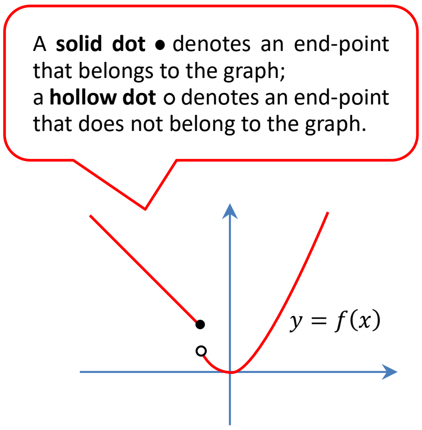
- Although the formula of $f \circ f$ can be written as $(f \circ f)(x) = x$, its domain is $\mathbb{R} \setminus \{0\}$ instead of the "natural domain" $\mathbb{R}$. There is a "hidden requirement" that $x \neq 0$.
- Although the formula of $f \circ g$ can be written as $(f \circ g)(x) = \frac{x-2}{x+1}$, its domain is $\mathbb{R} \setminus \{-1, 2\}$ instead of the "natural domain" $\mathbb{R} \setminus \{-1\}$. There is a "hidden requirement" that $x \neq 2$.
- Although the formula of $g \circ f$ can be written as $(g \circ f)(x) = \frac{1+x}{1-2x}$, its domain is $\mathbb{R} \setminus \left\{0, \frac{1}{2}\right\}$ instead of the "natural domain" $\mathbb{R} \setminus \left\{\frac{1}{2}\right\}$. There is a "hidden requirement" that $x \neq 0$.

We may also define a function by giving different rules of assignment to different parts of the domain. Functions defined in this way are called **piecewise-defined functions**.

Example 1.27 Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$

Evaluate $f(-2)$, $f(-1)$ and $f(0)$, and sketch the graph of f.



Solution:
- Since $-2 \leq -1$, we have $f(-2) = 1 - (-2) = 3$.
- Since $-1 \leq -1$, we have $f(-1) = 1 - (-1) = 2$.
- Since $0 > -1$, we have $f(0) = 0^2 = 0$.

The graph of f is sketched on the right.

Example 1.28 Piecewise defined functions are also common in our daily life. Suppose that we are taking an urban taxi ("red taxi") in Hong Kong. For the first 2 km traveled, the taxi fare is $\$29$; and for every subsequent 0.2 km, the fare is $\$2.1$ until the total amount reaches $\$102.5$, and is $\$1.4$ after the total amount has reached $\$102.5$. Let's try to express the total fare $\$P$ as a function of the total distance x km traveled. (The following fare model has been simplified.)

- If $0 < x \leq 2$, then the total fare is just $\$29$.
- If $x > 2$, then the total fare is $\$29$ plus the fare for the extra distance $(x - 2)$ km traveled, which is $\$2.1 \cdot \frac{x-2}{0.2}$ until the total amount reaches $\$102.5$. We observe that the total amount $\$29 + \$2.1 \cdot \frac{x-2}{0.2}$ reaches $\$102.5$ when $x = 9$, so this formula holds for $2 < x \leq 9$.
- If $x > 9$, i.e. after the total amount has reached $\$102.5$, then the total fare is $\$102.5$ plus the fare for the extra distance $(x - 9)$ km traveled, which is $\$1.4 \cdot \frac{x-9}{0.2}$.

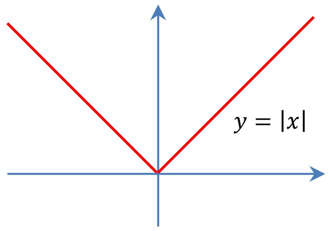
Therefore the function $P: (0, +\infty) \to \mathbb{R}$ is defined piecewise by

$$P(x) = \begin{cases} 29 & \text{if } 0 < x \leq 2 \\ 29 + 2.1 \cdot \dfrac{x-2}{0.2} & \text{if } 2 < x \leq 9 \\ 102.5 + 1.4 \cdot \dfrac{x-9}{0.2} & \text{if } x > 9 \end{cases} = \begin{cases} 29 & \text{if } 0 < x \leq 2 \\ 10.5x + 8 & \text{if } 2 < x \leq 9. \\ 7x + 39.5 & \text{if } x > 9 \end{cases}$$

A frequently used piecewise-defined function is the **absolute value function**.

Definition 1.29 The ***absolute value function*** is the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

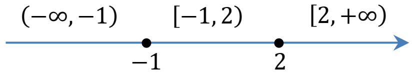


Example 1.30 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = |x+1| + |x-2|.$$

Rewrite the formula for f into the piecewise form.

Solution:



Note that $|x+1|$ and $|x-2|$ changes their form near -1 and near 2, so we divide $\mathbb{R}$ into the three intervals

$$(-\infty,-1), \qquad [-1,2) \qquad \text{and} \qquad [2,+\infty).$$

- If $x \in (-\infty,-1)$, then $x+1<0$ and $x-2<0$, so
$$|x+1| + |x-2| = -(x+1)-(x-2) = -2x+1.$$
- If $x \in [-1,2)$, then $x+1 \geq 0$ but $x-2<0$, so
$$|x+1| + |x-2| = (x+1)-(x-2) = 3.$$
- If $x \in [2,+\infty)$, then $x+1 \geq 0$ and $x-2 \geq 0$, so
$$|x+1| + |x-2| = (x+1)+(x-2) = 2x-1.$$

So in summary, we have

$$f(x) = \begin{cases} -2x+1 & \text{if } x \in (-\infty,-1) \\ 3 & \text{if } x \in [-1,2) \\ 2x-1 & \text{if } x \in [2,+\infty) \end{cases}.$$

Apart from operations on functions, we can also apply **transformations on the graph** of a function to obtain graphs of related functions. The following are six basic transformations on the graph of a function $\{(x,y): y = f(x)\}$, which correspond to actions on the variables x and y.

Theorem 1.31 *Let $k>0$, $c>1$, and the graph of a function f be given. Then we can obtain the graph of each of the following functions by doing the corresponding action on the graph of f:*

- $y = f(x) - k$ — *Shift the graph of $y = f(x)$ **downward** by k units*
- $y = f(x+k)$ — *Shift the graph of $y = f(x)$ to the **left** by k units*
- $y = \frac{1}{c} f(x)$ — ***Compress** the graph of $y = f(x)$ **vertically** by a factor of c*
- $y = f(cx)$ — ***Compress** the graph of $y = f(x)$ **horizontally** by a factor of c*
- $y = -f(x)$ — *Reflect the graph of $y = f(x)$ across the **x-axis***
- $y = f(-x)$ — *Reflect the graph of $y = f(x)$ across the **y-axis***

These basic transformations can be further composed with each other to produce other more complicated transformations.

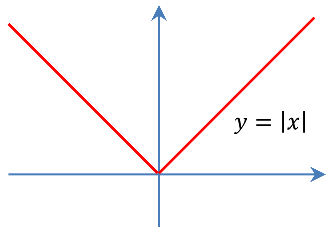
Remark 1.32 Let $k > 0$ and $c > 1$. To come up with Theorem 1.31 without much hard memorization, one can think of

- replacing y by $y + k$ as shifting **<u>the coordinate plane</u>** (NOT the graph) upward by k units,
- replacing x by $x + k$ as shifting **<u>the coordinate plane</u>** to the right by k units,
- replacing y by cy as stretching **<u>the coordinate plane</u>** vertically by a factor of c, and so on.

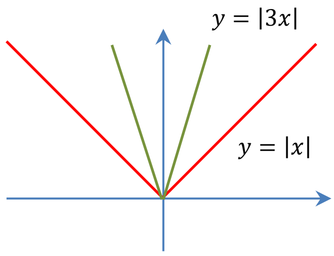
Example 1.33 Given the graph of $y = |x|$ on the right, sketch the graph of

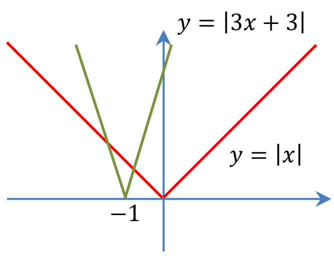
$$y = 4 - |3x + 3|$$

on the same coordinate plane.

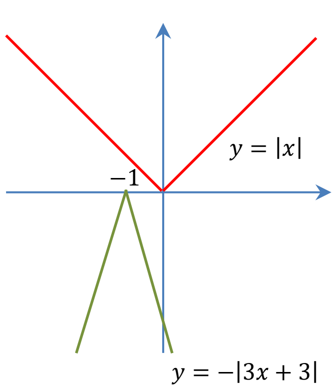


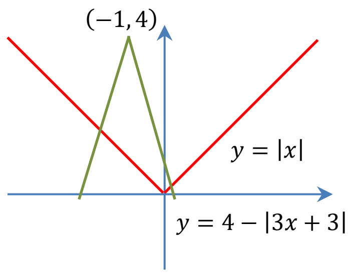
Solution:
We start with the graph of $y = |x|$. A horizontal compression by a factor of 3 gives the graph of $y = |3x|$, and a left translation of 1 unit gives the graph of $y = |3(x + 1)| = |3x + 3|$.





Next, a reflection across the x-axis gives the graph of $y = -|3x + 3|$, and an upward translation of 4 units gives the graph of $y = 4 - |3x + 3|$, which is the graph we want.





3. Properties of functions

Some functions have special **"symmetry" properties**.

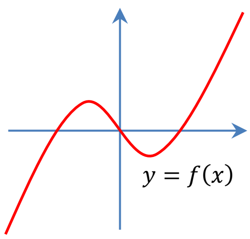
Definition 1.34 Let $f: \mathbb{R} \to \mathbb{R}$ be a function.

(i) f is an ***odd function*** if $f(-x) = -f(x)$ for every $x \in \mathbb{R}$;

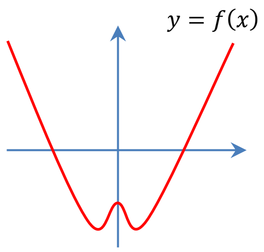
(ii) f is an ***even function*** if $f(-x) = f(x)$ for every $x \in \mathbb{R}$;

(iii) f is a ***periodic function*** if there exists $c > 0$ such that $f(x + c) = f(x)$ for every $x \in \mathbb{R}$. This positive number c is called a ***period*** of f. If there is a smallest such positive number c, then it is called the ***fundamental period*** of f.

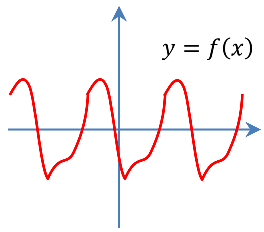
Remark 1.35 The graph of an odd function is **rotationally symmetric about the origin** by half a revolution. The graph of an even function is **symmetric across the y-axis**. The graph of a periodic function is **translationally symmetric along the x-axis**.



$f: \mathbb{R} \to \mathbb{R}$ is odd



$f: \mathbb{R} \to \mathbb{R}$ is even



$f: \mathbb{R} \to \mathbb{R}$ is periodic

Example 1.36 The simplest examples of odd and even functions are given by the monomials

$$f(x) = x^n$$

where n is a non-negative integer. f is an odd function if n is an odd integer, while f is an even function if n is an even integer.

Example 1.37 Let $f, g: \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x|x| \qquad \text{and} \qquad g(x) = x^2|x|.$$

Determine whether each of them is odd, and whether each of them is even.

Solution:

For every $x \in \mathbb{R}$, we have

$$f(-x) = (-x)|-x| = -x|x| = -f(x)$$

and

$$g(-x) = (-x)^2|-x| = x^2|x| = g(x).$$

So we conclude that f is an odd function and g is an even function.

Remark 1.38 Odd functions and even functions are **not** opposite concepts, unlike the same pair of terminology for integers. Some functions are both odd and even, while some are neither odd nor even. Can you construct examples for these situations?

Example 1.39 We will see in the next section of this chapter that the cosine and sine functions $\cos, \sin: \mathbb{R} \to \mathbb{R}$ are periodic functions. We have

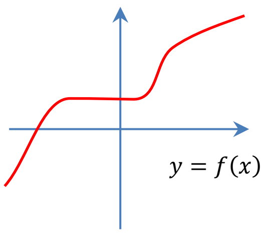
$$\cos(x + 2\pi) = \cos x \qquad \text{and} \qquad \sin(x + 2\pi) = \sin x \qquad \text{for every } x \in \mathbb{R},$$

so 2π is a period of these two functions. It turns out that 2π is also their fundamental period.

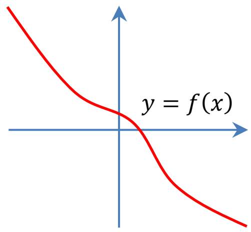
Some functions have special **"trend" properties**.

Definition 1.40 Let A be a non-empty set of real numbers. A function $f: A \to \mathbb{R}$ is said to be

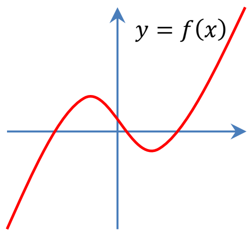
$$\left\{\begin{array}{c}\textit{(monotonic) increasing}\\ \textit{(monotonic) decreasing}\\ \textit{strictly increasing}\\ \textit{strictly decreasing}\end{array}\right\} \textit{\textbf{on } A} \text{ if } \left\{\begin{array}{c} f(a) \geq f(b)\\ f(a) \leq f(b)\\ f(a) > f(b)\\ f(a) < f(b)\end{array}\right\} \text{ for every pair of numbers } a > b \text{ in } A.$$



f is monotonic increasing on $\mathbb{R}$



f is strictly decreasing on $\mathbb{R}$



f is not monotone on $\mathbb{R}$

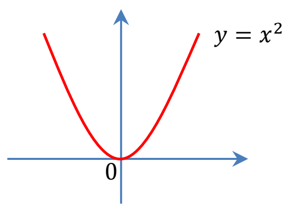
Example 1.41 Let $f: \mathbb{R} \to \mathbb{R}$ be the monomial

$$f(x) = x^2.$$

Then for every **non-negative** real numbers $a > b$, we have

$$f(a) = a^2 > b^2 = f(b),$$

so f is **strictly increasing on** $[0, +\infty)$.



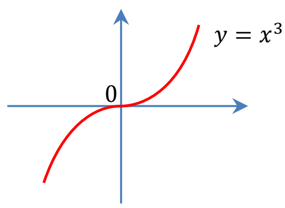
Example 1.42 Let $f: \mathbb{R} \to \mathbb{R}$ be the monomial

$$f(x) = x^3.$$

Then for **every real numbers** $a > b$, we have

$$f(a) = a^3 > b^3 = f(b),$$

so f is **strictly increasing on** $\mathbb{R}$.



Example 1.43 In general, let n be a positive integer and let $f: \mathbb{R} \to \mathbb{R}$ be the monomial

$$f(x) = x^n.$$

- If n is odd, then f is strictly increasing on $\mathbb{R}$.
- If n is even, then f is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.

Example 1.44 Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function

$$f(x) = \frac{1}{x}.$$

Then f is strictly decreasing on $(0, +\infty)$, and f is strictly decreasing on $(-\infty, 0)$ as well.

Think: Does this imply that f is strictly decreasing on $\mathbb{R} \setminus \{0\}$?
[No! We have $1 > -1$ and $f(1) > f(-1)$.]

Example 1.45 Let $f: \mathbb{R} \to \mathbb{R}$ be the absolute value function

$$f(x) = |x|.$$

Then f is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.

Some functions have special **"boundedness" properties**.

Definition 1.46 Let A be a set and $f: A \to \mathbb{R}$ be a function.

(i) If there exists a real number M such that

$$f(x) \le M \qquad \text{for every } x \in A,$$

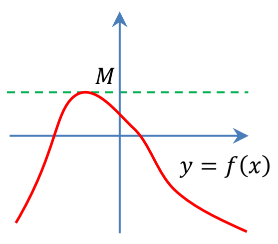
then f is said to be ***bounded from above on A*** and M is called an ***upper bound*** of f on A.

(ii) If there exists a real number m such that

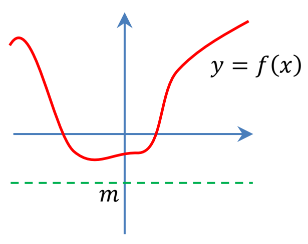
$$f(x) \ge m \qquad \text{for every } x \in A,$$

then f is said to be ***bounded from below on A*** and m is called a ***lower bound*** of f on A.

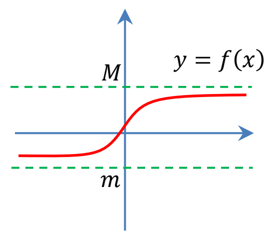
(iii) If f is both bounded from above and from below on A, then f is said to be ***bounded on A***.



$f: \mathbb{R} \to \mathbb{R}$ is bounded from above



$f: \mathbb{R} \to \mathbb{R}$ is bounded from below



$f: \mathbb{R} \to \mathbb{R}$ is bounded

Example 1.47 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = x^2.$$

Then f is not bounded from above, but is bounded from below by 0. Note that f is also bounded from below by every negative number.

Example 1.48 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \frac{1}{x^2 + 2x + 2}.$$

Note that we can rewrite the denominator to obtain

$$f(x) = \frac{1}{(x+1)^2 + 1}.$$

Since $(x+1)^2 + 1 \geq 1$ for every $x \in \mathbb{R}$, it follows that

$$0 < f(x) \leq 1 \qquad \text{for every } x \in \mathbb{R}.$$

So f is bounded from above by 1 and bounded from below by 0. In other words, f is a bounded function.

Some functions have special **"mapping" properties**.

Definition 1.49 Let A and B be sets and $f: A \to B$ be a function. f is said to be ***one-to-one*** (or ***injective***) if for every $a, b \in A$,

$$a \neq b \qquad \text{implies} \qquad f(a) \neq f(b),$$

i.e. different elements in A take different values in B. Note that this is the same as requiring that for every $a, b \in A$,

$$f(a) = f(b) \qquad \text{implies} \qquad a = b.$$

Example 1.50 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

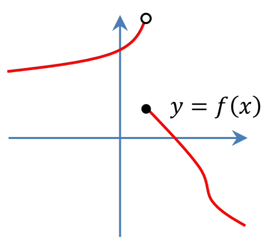
$$f(x) = 2x + 1.$$

For every $a, b \in \mathbb{R}$, if we assume that $f(a) = f(b)$, then

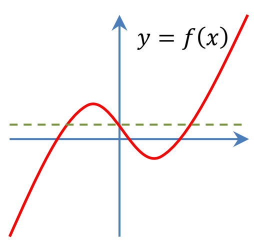
$$\begin{aligned} \Rightarrow \quad 2a + 1 &= 2b + 1 \\ \Rightarrow \quad 2a &= 2b \\ \Rightarrow \quad a &= b. \end{aligned}$$

This shows that f is one-to-one.

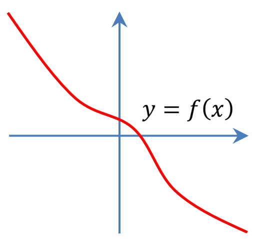
Remark 1.51 ("Horizontal line test") A function f is one-to-one if and only if every **horizontal line** in the coordinate plane intersects the graph of f at **no more than one point**.



$f: \mathbb{R} \to \mathbb{R}$ is one-to-one



$f: \mathbb{R} \to \mathbb{R}$ is not one-to-one



$f: \mathbb{R} \to \mathbb{R}$ is one-to-one

Example 1.52 Every strictly increasing function is one-to-one. Every strictly decreasing function is also one-to-one.

Example 1.53 For each of the following functions, determine whether it is one-to-one.

(a) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

(b) $g: [0, +\infty) \to \mathbb{R}$ defined by $g(x) = x^2$

Solution:

(a) Since 1 and -1 are two different real numbers with

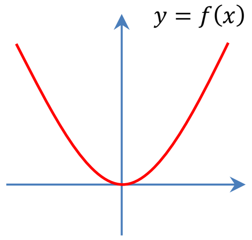
$$f(1) = 1 = f(-1),$$

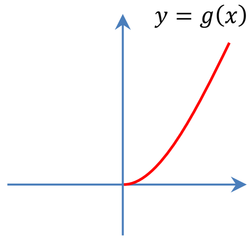
we see that f is not one-to-one.

(b) Suppose that a and b are **non-negative real numbers** such that $g(a) = g(b)$. Then

$$a^2 = b^2.$$

Taking positive square roots on both sides, we get $a = b$; so g is one-to-one.





Why does the same reasoning (taking positive square roots) fail for the function f in (a)?

Remark 1.54 From Example 1.53, we observe that whether a function is **one-to-one or not** depends very much on its **domain**. If a function is not one-to-one, it is sometimes possible to make it one-to-one by modifying its domain to a smaller set.

One-to-one functions are important because they have **inverses**. An inverse can be understood as a "reverse machine" which undoes everything the given function does.

Theorem 1.55 *Let f be a one-to-one function whose domain is A and whose range is B. Then there exists a unique function $g: B \to A$ such that*

$$(f \circ g)(y) = y \quad \text{for every } y \in B \qquad \text{and} \qquad (g \circ f)(x) = x \quad \text{for every } x \in A.$$

Sloppy proof. Since B is the range of the one-to-one function f, for each $y \in B$, the equation

$$f(x) = y$$

has one and only one solution $x \in A$. Now we define the function $g: B \to A$ by

$$g(y) := x.$$

Then this g is the only function satisfying the required properties. ■

Definition 1.56 Let f be a one-to-one function. Then the function g constructed as in Theorem 1.55 is called the ***inverse*** of f, and we denote it as f^{-1}.

Note that f^{-1} does NOT mean $\frac{1}{f}$.

Example 1.50 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = 2x + 1.$$

We have already seen that f is one-to-one. Now we note that the range of f is the whole $\mathbb{R}$, and let $g: \mathbb{R} \to \mathbb{R}$ be the function

$$g(x) = \frac{x-1}{2}.$$

Then for every $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x, \qquad \text{and}$$

$$(g \circ f)(x) = g(2x+1) = \frac{(2x+1)-1}{2} = x.$$

So g is the inverse of f, i.e. $f^{-1}: \mathbb{R} \to \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x-1}{2}.$$

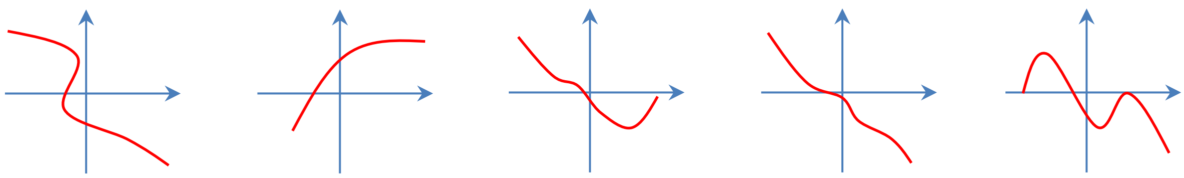
Remark 1.57 Given a **one-to-one** function f, Theorem 1.55 implies that

- ⦿ the domain of f^{-1} is the range of f, and
- ⦿ the range of f^{-1} is the domain of f.

We also have

- ⦿ $f\big(f^{-1}(y)\big) = y$ for every y in the range of f, and
- ⦿ $f^{-1}\big(f(x)\big) = x$ for every x in the domain of f.

Example 1.58 Which of the following shows the graph of a function that has an inverse throughout its domain?



(a) (b) (c) (d) (e)

Solution:

We look for the graph of a one-to-one function. Such a graph must intersect each vertical line and each horizontal line in the coordinate plane at one point at most. Only (d) satisfies this requirement.

Remark 1.59 To find the inverse of a given one-to-one function f, we do the following three steps:

(i) For each y in the range of f, we write $y = f(x)$.

(ii) Find a solution x (in the domain of f) to this equation in terms of y. This is $f^{-1}(y)$.

(iii) Rename the input as x instead of y.

Example 1.60 Let $f: (-\infty, 0] \to \mathbb{R}$ be the function defined by

$$f(x) = x^2.$$

Find the inverse of f and sketch the graphs of f and of f^{-1} on the same coordinate plane.

Solution:

The range of f is $[0, +\infty)$. Writing $y = f(x)$ with $y \in [0, +\infty)$, our aim is to find a solution for x **in the domain** $(-\infty, \mathbf{0}]$ **of** $\boldsymbol{f}$:

$$y = x^2 \qquad \Rightarrow \qquad x = -\sqrt{y}.$$

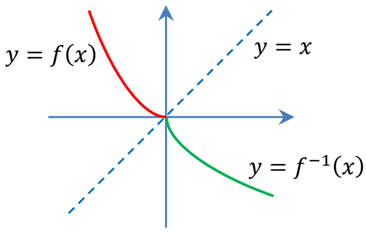
Therefore we get

$$f^{-1}(y) = -\sqrt{y}.$$

We usually prefer to use the symbol x (instead of y) to denote the input of a function, so the inverse $f^{-1}: [0, +\infty) \to \mathbb{R}$ is given by

$$f^{-1}(x) = -\sqrt{x}.$$

The graphs of f and f^{-1} are shown below.



Remark 1.61 Suppose that the graph of a one-to-one function f is given. Then the graph of f^{-1} is obtained by reflecting the graph of f across the line with slope 1 that passes through the origin.

Theorem 1.62 *Let f and g be one-to-one functions. Then $f \circ g$ is also one-to-one and*

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

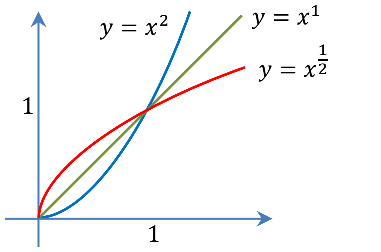
Sloppy proof. Recall that $(f \circ g)(x) = f\big(g(x)\big)$. Now given an output $f\big(g(x)\big)$, in order to undo the process done by $f \circ g$, we <u>first apply f^{-1}</u> to it so as to get $g(x)$, and <u>then further apply g^{-1}</u> to it so as to recover the original input x. Symbolically we have

$$(g^{-1} \circ f^{-1})\big((f \circ g)(x)\big) = g^{-1}\Big(f^{-1}\Big(f\big(g(x)\big)\Big)\Big) = g^{-1}\big(g(x)\big) = x.$$

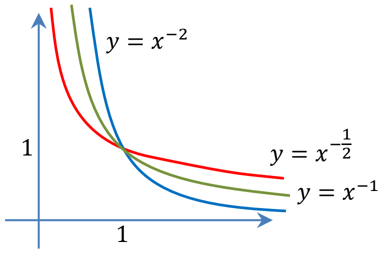
This means that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. ■

4. Elementary functions

Apart from **polynomials** and **rational functions** which have been introduced in the previous sections, there are still many other types of elementary functions which are useful in our daily lives.



Definition 1.63 Let a be a real number. Then the function $f(x) = x^a$ is called a ***power function***.



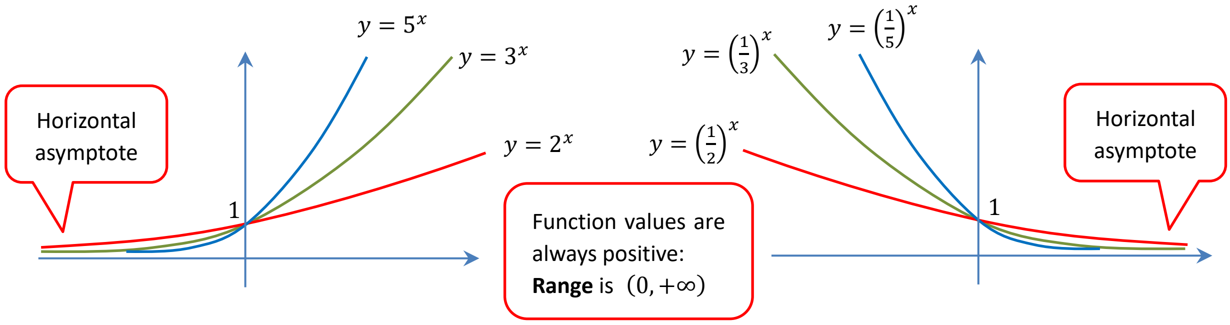
Some important special cases of power functions include

- monomials (when a is a non-negative integer),
- the n^{th} root function (when $a = 1/n$ and n is a positive integer), and
- the reciprocal (when $a = -1$).

Think: What are the domain and the range of a power function?

Definition 1.64 Let $a > 0$ and $a \neq 1$. The function $f(x) = a^x$ is called the ***exponential function with base a***. (When $a = 1$ this is just a boring constant function.)

Let's compare the graphs of the various exponential functions for $0 < a < 1$ and for $a > 1$:



Lemma 1.65 *For each $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is a <u>one-to-one</u> function with <u>domain $\mathbb{R}$</u> and <u>range $(0, +\infty)$</u>.*

Remark 1.66 **Do not mix up the power functions and the exponential functions!** The input x in a power function x^a is located at the base, while in an exponential function a^x it is located at the exponent.

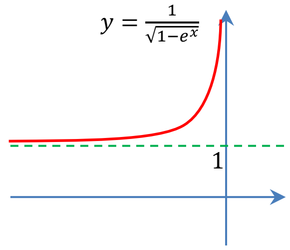
Definition 1.67 In calculus we will often encounter a special base $e \approx 2.71828$, which is an irrational number. The particular exponential function $f(x) = e^x$ with base e is called the ***natural exponential function*** and is sometimes denoted also as $f(x) = \exp x$.

Example 1.68 Find the natural domain and the range of the function

$$f(x) = \frac{1}{\sqrt{1-e^x}}.$$

Solution:

The expression $\frac{1}{\sqrt{1-e^x}}$ is well-defined if and only if $1 - e^x > 0$, which holds if and only if $x < 0$; so the natural domain of f is $(-\infty, 0)$. On the other hand, setting $y = \frac{1}{\sqrt{1-e^x}}$ for $x \in (-\infty, 0)$:



- We obviously have $y > 0$;
- Rearranging the equation we also get

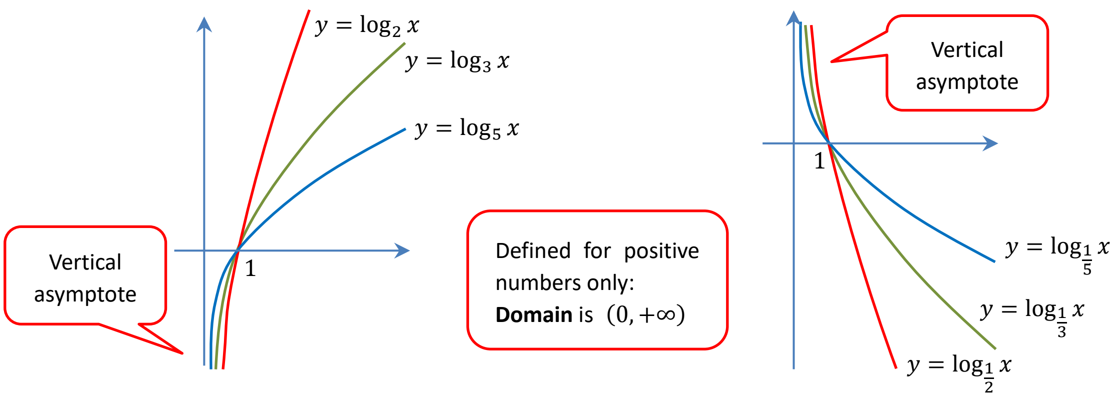
$$e^x = 1 - \frac{1}{y^2},$$

which implies that $1 - \frac{1}{y^2} > 0$, i.e. $y > 1$ or $y < -1$.

Combining these two requirements we have $y > 1$, so the range of f is $(1, +\infty)$.

Definition 1.69 Let $a > 0$ and $a \neq 1$. The inverse of the exponential function with base a is called the ***logarithmic function with base*** $\boldsymbol{a}$ and is denoted by $f(x) = \log_a x$.

As expected, we can obtain the graph of the logarithmic function $y = \log_a x$ by reflecting the graph of the corresponding exponential function $y = a^x$ across the line $y = x$.



Lemma 1.70 *For each $a > 0$ and $a \neq 1$, the logarithmic function $f(x) = \log_a x$ is a* <u>*one-to-one*</u> *function with* <u>*domain* $(0, +\infty)$</u> *and* <u>*range* $\mathbb{R}$</u>.

Definition 1.71 The logarithmic function $f(x) = \log_e x$ with the special base $e \approx 2.71828$ is called the ***natural logarithmic function*** and is commonly denoted by $f(x) = \ln x$.

Since there is an **inverse relation** between the exponential and logarithmic functions (with the same base), we obtain the following relations from Theorem 1.55.

Proposition 1.72 *Let* a *be a positive number with* $a \neq 1$. *Then*
(i) $a^{\log_a x} = x$ *and in particular* $e^{\ln x} = x$ *for every* $x > 0$,
(ii) $\log_a a^x = x$ *and in particular* $\ln e^x = x$ *for every* $x \in \mathbb{R}$.

Remark 1.73 Do remember the **laws of indices** and the **laws of logarithms** in your computations.

Laws of indices	Laws of logarithms
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^{x+y} = a^x a^y$	$\log_a x + \log_a y = \log_a xy$
$a^{x-y} = \dfrac{a^x}{a^y}$	$\log_a x - \log_a y = \log_a \dfrac{x}{y}$
$a^{xy} = (a^x)^y$	$\log_a x^y = y \log_a x$
$a^x b^x = (ab)^x$	$\log_a b = \dfrac{\log_c b}{\log_c a}$
$\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$	

These formulas hold whenever **both sides are well-defined**.

Example 1.74 To provide land for housing, the government of city A now decides to develop 5% of its country park into land for housing every two years in the future. How long does it take to diminish half of the country park in city A?

Solution:
Let the current area of the country park in city A be C. The area $A(t)$ of country park remaining after t years is given by the exponential function

$$A(t) = C(1 - 5\%)^{\frac{t}{2}} = C(0.95)^{\frac{t}{2}}.$$

When half of the country park is diminished, we have

$$A(t) = \frac{1}{2}C,$$

so solving the equation

$$\frac{1}{2}C = C(0.95)^{\frac{t}{2}}$$

we obtain

$$t = 2 \cdot \frac{\ln 0.5}{\ln 0.95} \approx 27.027.$$

Therefore it takes about 27.027 years for half of the country park to diminish.

Example 1.75 Find all the real numbers x which satisfies the equation

$$\log_{2x}(2x^2 + 2x + 12) = 2.$$

Solution:

Our first aim is to equate the bases of logarithm on both sides, so we write

$$2 = 2\log_{2x} 2x = \log_{2x}(2x)^2.$$

Then the given equation becomes

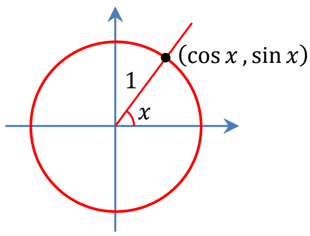
$$\begin{aligned} & \log_{2x}(2x^2 + 2x + 12) = \log_{2x}(2x)^2 \\ \Rightarrow \quad & 2x^2 + 2x + 12 = (2x)^2 \\ \Rightarrow \quad & -2x^2 + 2x + 12 = 0 \\ \Rightarrow \quad & -2(x-3)(x+2) = 0 \end{aligned}$$

So $x = 3$ or $x = -2$. However, since $2x$ was the base of a logarithm, we must require $2x > 0$. Therefore $x = -2$ is not a solution to the equation, and the only valid solution is

$$x = 3.$$

Now we define the six trigonometric functions.

Definition 1.76 The ***cosine function*** $f(x) = \cos x$ and the ***sine function*** $g(x) = \sin x$ are defined on $\mathbb{R}$ as follows. For each real number x, we let P be the point of intersection of the unit circle centered at the origin and the ray emanating from the origin making an angle of x radians from the positive horizontal axis measured counterclockwise. Then $\cos x$ is defined to be the first coordinate of P and $\sin x$ is defined to be the second coordinate of P.



Remark 1.77 **Always use "radians" and never use "degrees"** as the unit of the size of an angle in this course. This will help avoid troubles in calculus computations. Bearing in mind that the unit circle has circumference 2π, we can convert between the degree measure and the radian measure of an angle using the formulas

$$x^\circ = \frac{2\pi}{360^\circ}x^\circ = \frac{\pi x}{180} \qquad \text{and} \qquad x = \frac{360^\circ}{2\pi}x = \left(\frac{180x}{\pi}\right)^\circ.$$

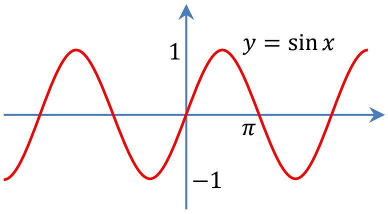
Definition 1.78 The ***tangent function*** is defined to be the quotient of the sine by the cosine, i.e.

$$\tan x \coloneqq \frac{\sin x}{\cos x}.$$

We also define the ***cotangent***, the ***secant*** and the ***cosecant functions*** respectively by

$$\cot x \coloneqq \frac{\cos x}{\sin x}, \qquad \sec x \coloneqq \frac{1}{\cos x} \qquad \text{and} \qquad \csc x \coloneqq \frac{1}{\sin x}.$$

Remark 1.79 The following are the graphs of the three basic trigonometric functions and some important facts. Try to sketch the graphs of cot, sec and csc on your own.



Domain: $\mathbb{R}$

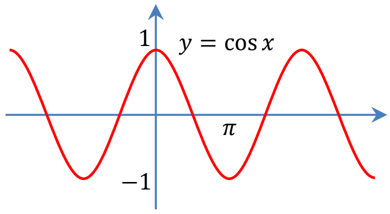
Range: $[-1, 1]$

Fundamental **period**: 2π

Parity: **Odd**

Special values:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$



Domain: $\mathbb{R}$

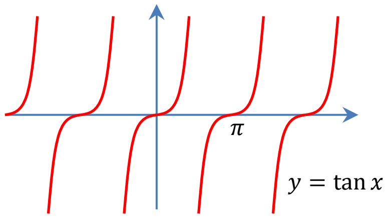
Range: $[-1, 1]$

Fundamental **period**: 2π

Parity: **Even**

Special values:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$



Domain: $\mathbb{R} \setminus \left\{\left(n + \frac{1}{2}\right)\pi : n \text{ is an integer}\right\}$

Range: $\mathbb{R}$

Fundamental **period**: π

Parity: **Odd**

Special values:

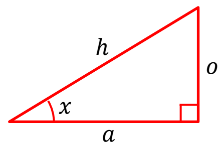
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

Remark 1.80 The following are some important **formulas in trigonometry**. Please make yourself familiar with all these results which you should have learnt from high school already.

Right-angled triangle

$$\cos x = \frac{a}{h} \qquad \tan x = \frac{o}{a} \qquad \sec x = \frac{h}{a}$$

$$\sin x = \frac{o}{h} \qquad \cot x = \frac{a}{o} \qquad \csc x = \frac{h}{o}$$



Pythagorean identities

$$\cos^2 x + \sin^2 x = 1$$

Divide both sides by $\cos^2 x$ Divide both sides by $\sin^2 x$

$$1 + \tan^2 x = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

Phase-shift identities

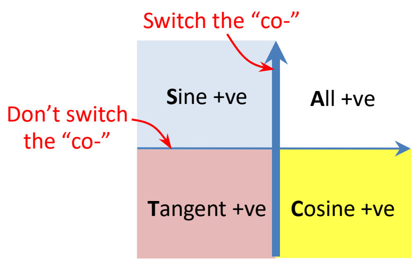
Use the **A-S-T-C diagram**, for example:

$$\sin(\pi - x) = \sin x \qquad \sec(-x) = \sec x$$

$$\cos\left(\frac{3\pi}{2} - x\right) = -\sin x \qquad \csc\left(\frac{5\pi}{2} - x\right) = \sec x$$

$$\tan\left(\frac{\pi}{2} + x\right) = -\cot x \qquad \cot\left(\frac{3\pi}{2} + x\right) = -\tan x$$

and so on.



"**A**ll **S**tudents **T**ake **C**alculus"

Addition of angles (a.k.a. Compound angle formulae)

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Oscillatory motion

Let $r > 0$, $\omega \neq 0$ and $\theta \in \mathbb{R}$ be constants. The function

$$f(t) = r\sin(\omega t + \theta)$$

describes the position at time t of a moving particle performing an **oscillatory motion** along a line.

This oscillation has **amplitude** r, **frequency** $\frac{|\omega|}{2\pi}$ and **fundamental period** $\frac{2\pi}{|\omega|}$.

Addition of sinusoids (a.k.a. Subsidiary angle form)

Any sum of constant multiples of $\sin \omega t$ and $\cos \omega t$ can be written in the form $r\sin(\omega t + \theta)$ or $r\cos(\omega t + \theta)$: If a and b are not both zero, then

$$a \sin \omega t + b \cos \omega t = \underbrace{\sqrt{a^2 + b^2}}_{r}\left(\sin \omega t \cdot \underbrace{\frac{a}{\sqrt{a^2 + b^2}}}_{\cos\theta} + \cos \omega t \cdot \underbrace{\frac{b}{\sqrt{a^2 + b^2}}}_{\sin\theta}\right)$$
$$= r\sin(\omega t + \theta)\,,$$

where $r = \sqrt{a^2 + b^2}$, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$.

Sum-to-product / Product-to-sum formulae

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

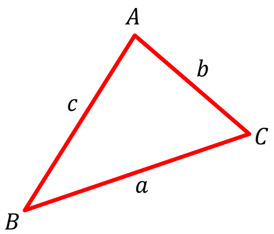
$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

Cosine Law and Sine Law

In a triangle ΔABC lying in a plane, let the lengths of the edges BC, AC and AB be a, b and c respectively. Then

(i) (Cosine Law) $c^2 = a^2 + b^2 - 2ab\cos C$, and

(ii) (Sine Law) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



Example 1.81 Prove the following identities:

(a) $\sin^2\left(x-\frac{3\pi}{2}\right)+\cos^2\left(\frac{\pi}{2}+x\right)+\tan^2(x-2\pi)=\sec^2 x$

(b) $\cos x\cos 2x\cos 3x=\frac{1}{4}(1+\cos 2x+\cos 4x+\cos 6x)$

Proof:

(a) Whenever each term is well-defined, we have

$$\begin{aligned}\sin^2\left(x-\frac{3\pi}{2}\right)+\cos^2\left(\frac{\pi}{2}+x\right)+\tan^2(x-2\pi) &= \cos^2 x+\sin^2 x+\tan^2 x\\ &= 1+\tan^2 x\\ &= \sec^2 x.\end{aligned}$$

(b) For every real number x, we have

$$\begin{aligned}\cos x\cos 2x\cos 3x &= \frac{1}{2}(\cos 3x+\cos x)\cos 3x\\ &= \frac{1}{2}\cos^2 3x+\frac{1}{2}\cos x\cos 3x\\ &= \frac{1}{2}\left(\frac{1+\cos 6x}{2}\right)+\frac{1}{4}(\cos 4x+\cos 2x)\\ &= \frac{1}{4}+\frac{1}{4}\cos 2x+\frac{1}{4}\cos 4x+\frac{1}{4}\cos 6x\\ &= \frac{1}{4}(1+\cos 2x+\cos 4x+\cos 6x).\end{aligned}$$

Example 1.82 Find all the real numbers x which satisfy the equation

$$\cos x-\sqrt{3}\sin x=1.$$

Solution:

$$\begin{aligned} & \cos x-\sqrt{3}\sin x=1\\ \Rightarrow\quad & 2\left(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}\right)=1\\ \Rightarrow\quad & \cos\left(x+\frac{\pi}{3}\right)=\frac{1}{2}\end{aligned}$$

This gives $x+\frac{\pi}{3}=2n\pi+\frac{\pi}{3}$ or $x+\frac{\pi}{3}=2n\pi-\frac{\pi}{3}$, where n is an integer. Thus the solutions are

$$x=2n\pi \qquad \text{or} \qquad x=2n\pi-\frac{2\pi}{3},$$

where n is an integer.

Example 1.83 The exact value of $\sin\frac{5\pi}{12}$ is given by

$$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Example 1.84 The position of an oscillating particle at time t is given by

$$x(t) = 3\cos 2t - 4\sin 2t\,.$$

Find the amplitude and the fundamental period of this oscillatory motion.

Solution:

The position function can be rewritten as

$$x(t) = 3\cos 2t - 4\sin 2t = \sqrt{3^2+4^2}\left(\cos 2t\cdot\frac{3}{\sqrt{3^2+4^2}} - \sin 2t\cdot\frac{4}{\sqrt{3^2+4^2}}\right)$$
$$= 5\cos(2t+\theta)\,,$$

where θ is some real number such that $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. The amplitude of this oscillatory motion is 5, and the fundamental period is $2\pi/2 = \pi$.

The trigonometric functions are not one-to-one functions on their natural domains, so in order to define their inverses, we need to **restrict their domains** so that they become **one-to-one**. We want to choose domains that are large enough so that the ranges of these functions are not affected. Conventionally we choose the domains $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for $\sin$, $[0,\pi]$ for $\cos$, and $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for $\tan$.

Definition 1.85 The ***inverse trigonometric functions*** are defined as follows.

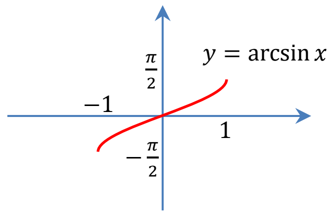
- The cosine function is one-to-one when its domain is restricted to $[0,\pi]$, and its inverse is denoted by $\arccos\colon[-1,1]\to[0,\pi]$.
- The sine function is one-to-one when its domain is restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and its inverse is denoted by $\arcsin\colon[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- The tangent function is one-to-one when its domain is restricted to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and its inverse is denoted by $\arctan\colon\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
- The functions arccot, arcsec and arccsc are defined in a similar manner.

Remark 1.86 In some textbooks, the inverse trigonometric functions are denoted as $\cos^{-1}$, $\sin^{-1}$ and $\tan^{-1}$. Although these notations are also used commonly, they may lead to confusions. For instance we have $\sin^2 x \equiv (\sin x)^2$, $\sin^3 x \equiv (\sin x)^3$ but

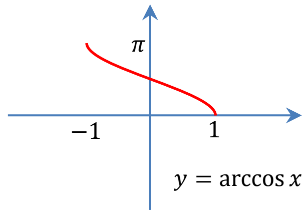
$$\sin^{-1} x \not\equiv (\sin x)^{-1}$$

because the left-hand side means $\arcsin x$ but the right-hand side means $\csc x$.

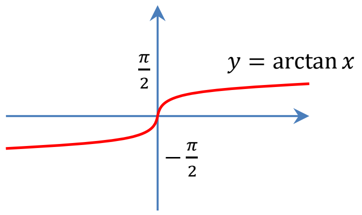
The graphs of the inverse trigonometric functions can be obtained by reflecting **the chosen portion** of the graph of the corresponding trigonometric function across the line $y = x$:



Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



Domain: $[-1, 1]$
Range: $[0, \pi]$



Domain: $\mathbb{R}$
Range: $(-\pi/2, \pi/2)$

Proposition 1.87 *We have the following inverse relations:*

(i) $\arcsin(\sin x) = x$ *for every* $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *and* $\sin(\arcsin x) = x$ *for every* $x \in [-1, 1]$.

(ii) $\arccos(\cos x) = x$ *for every* $x \in [0, \pi]$, *and* $\cos(\arccos x) = x$ *for every* $x \in [-1, 1]$.

(iii) $\arctan(\tan x) = x$ *for every* $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *and* $\tan(\arctan x) = x$ *for every* $x \in \mathbb{R}$.

Example 1.88 Without using a calculator, find the value of each of the following:

(a) $\arcsin\left(\sin\dfrac{6\pi}{5}\right)$ (b) $\tan\left(\arcsin\dfrac{1}{3}\right)$

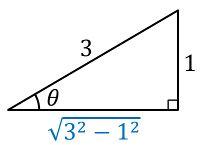
Solution:

(a) Note that $\sin\frac{6\pi}{5} = \sin\left(\pi - \frac{6\pi}{5}\right) = \sin\left(-\frac{\pi}{5}\right)$ and the number $-\frac{\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus

$$\arcsin\left(\sin\frac{6\pi}{5}\right) = \arcsin\left[\sin\left(-\frac{6\pi}{5}\right)\right] = -\frac{\pi}{5}.$$

(b) Let $\theta = \arcsin\frac{1}{3}$. Then $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\sin\theta = \frac{1}{3}$, so

$$\tan\left(\arcsin\frac{1}{3}\right) = \tan\theta = \frac{1}{\sqrt{3^2 - 1^2}} = \frac{1}{\sqrt{8}}.$$



All the functions that we are going to study in this course will basically come from this catalog of elementary functions in Section 1.4, as well as all those functions that we can construct from them using the methods introduced in Section 1.2.

Summary of Chapter 1

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ Basic notions about functions: **Domain**, **codomain**, **range**
- ✓ The **graph** of a function, "vertical line test"

- ✓ **Arithmetic operations** on functions
  - ⦿ $+$, $-$, $\times$, $\div$
  - ⦿ Composition $\circ$
- ✓ **Piecewise defined** functions
  - ⦿ The absolute value function
- ✓ **Transformations** of graphs of functions
  - ⦿ Vertical and horizontal **translation**
  - ⦿ Vertical and horizontal **compression / stretch**
  - ⦿ **Reflection** across the x-axis or the y-axis

- ✓ Properties of functions
  - ⦿ **Odd** functions / **Even** functions
  - ⦿ **Periodic** functions
  - ⦿ (Strictly) **increasing** functions / (Strictly) **decreasing** functions
  - ⦿ **Bounded** functions
  - ⦿ **One-to-one** functions, "horizontal line test"
- ✓ The **inverse** of a one-to-one function and its graph

- ✓ Graphs and properties of **elementary functions**:
  - ⦿ **Polynomials**: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
  - ⦿ **Rational** functions: Quotients of polynomials
  - ⦿ **Power** functions: x^a
  - ⦿ **Exponential** functions: a^x, e^x
  - ⦿ **Logarithmic** functions: $\log_a x$, $\ln x$
  - ⦿ **Trigonometric** functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$
  - ⦿ **Inverse trigonometric** functions: $\arcsin x$, $\arccos x$, $\arctan x$, etc.